INFORMATION AND CONTROL 44, 36–46 (1980)

Congruences Defined by Languages and Filters

HELMUT PRODINGER

Institut für Mathematische Logik und Formale Sprachen, TU Wien, Gusshausstrasse 27–29, A-1040, Wien, Austria

The usual right congruence $\sim_L$ can be generalized in the following manner: $x \sim_{\mathscr{L},L} y :\Leftrightarrow \{z \mid xz \in L \Leftrightarrow yz \in L\} \in \mathscr{L}$, where $\mathscr{L}$ is a family of languages. It turns out to be useful when $\mathscr{L}$ is a filter with an additional property. Furthermore semifilters are introduced and studied. It is also possible to define congruences by filters. Assuming the (right) congruences to have finite index yields a generalization of the regular sets.

1. INTRODUCTION AND PRELIMINARIES

The well-known mathematical concept *filter* has been already used in the theory of formal languages (Benda, Bendová, 1976). The same will be done here, but the point of view is another one: Let $L \subseteq \Sigma^*$ and let $\mathscr{L} \subseteq \mathfrak{P}(\Sigma^*)$ be a filter with a certain division property (see below). Then by

$$x \sim_{\mathscr{L},L} y :\Leftrightarrow \{z \mid xz \in L \Leftrightarrow yz \in L\} \in \mathscr{L}$$

a right congruence is defined, which reduces to the well-known right congruence $\sim_L$ of the theory of formal languages by taking $\mathscr{L} = \{\Sigma^*\}$.

A similar concept is used in model theory. (See Bell, Machover (1977, p. 174 ff.).)

With respect to the use of systems $\mathscr{L} \subseteq \mathfrak{P}(\Sigma^*)$ in the theory of formal languages compare also (Prodinger, Urbanek, 1979) and (Prodinger, 1979).

In Section 2 necessary and sufficient conditions for a family $\mathscr{L}$ are presented to define a right congruence; appropriate definitions will be given.

In Section 3 the concepts introduced in Section 2 are investigated in detail.

In Section 4 the considerations are extended to the case of congruences.

In Section 5 some generalizations of the family of the regular sets are introduced and closure properties of these families are investigated.

In Section 6 some remarks are made concerning probably the most interesting special case (i.e., if $\mathscr{L}$ is the family of cofinite sets).

Now the essential definitions are given: Σ^* denotes the free monoid generated by Σ with unit ϵ, $\Sigma^+ = \Sigma^* - \{\epsilon\}$. $\triangle$ denotes the symmetrical difference of two sets; $A \circ B := (A \triangle B)^c$. $w\backslash L = \{z \mid wz \in L\}$ and $L/w = \{z \mid zw \in L\}$.

0019-9958/80/010036-11$02.00/0

For a formal language L let

$$G_L(x, y) := \{z \mid xz \in L \Leftrightarrow yz \in L\} = (x\backslash L) \circ (y\backslash L).$$

The right congruence $\sim_L$ is defined by

$$x \sim_L y :\Leftrightarrow G_L(x, y) = \Sigma^*.$$

Finite automata are written as quintuples $(Q, \Sigma, \delta, q_0, F)$. If no final states are considered it will be written (Q, Σ, δ, q_0). The termini state and class are used synonymously. (The state q corresponds to the class $\{w \in \Sigma^* \mid \delta(q_0, w) = q\}$.)

If there is said nothing else, it is assumed that an arbitrary but fixed alphabet Σ is given.

It is to be remarked that this paper allows a family $\mathscr{L}$ to be empty.

Concepts of the theory of formal languages not especially described can be found in (Eilenberg, 1974).

2. Right Congruences and Filters

Definition 2.1. A family of languages $\mathscr{L} \subseteq \mathfrak{P}(\Sigma^*)$ is called a *filter with division property* (FD), if the following axioms are valid:

(FD1) $\mathscr{L} \neq \varnothing$

(FD2) $A, B \in \mathscr{L} \Rightarrow A \cap B \in \mathscr{L}$

(FD3) $A \in \mathscr{L},\ A \subseteq B \Rightarrow B \in \mathscr{L}$

(FD4) $A \in \mathscr{L},\ z \in \Sigma^* \Rightarrow z\backslash A \in \mathscr{L}$.

Definition 2.2. A family of languages $\mathscr{L} \subseteq \mathfrak{P}(\Sigma^*)$ is called a *semifilter with division property* (SFD), if the following axioms are valid:

(SFD1) $\Sigma^* \in \mathscr{L}$

(SFD2) $A, B \in \mathscr{L} \Rightarrow A \circ B \in \mathscr{L}$

(SFD3) $A \in \mathscr{L},\ z \in \Sigma^* \Rightarrow z\backslash A \in \mathscr{L}$.

Each FD is also an SFD: (SFD1) follows from (FD1) and (FD3); (SFD2) follows from (FD2) and (FD3) if $A \circ B = (A \cap B) \cup (A \cup B)^c$ is taken in account.

Definition 2.3. $x \sim_{\mathscr{L},L} y :\Leftrightarrow G_L(x, y) \in \mathscr{L}$.

Theorem 2.4. *If $\mathscr{L}$ is an* SFD *then $\sim_{\mathscr{L},L}$ is a right congruence.*

Proof. The reflexivity follows from (SFD1). The symmetry is clear. The transitivity can be seen as follows:

$$G_L(x, z) = (x\backslash L) \circ (z\backslash L) = (x\backslash L) \circ \Sigma^* \circ (z\backslash L)$$
$$= (x\backslash L) \circ (y\backslash L) \circ (y\backslash L) \circ (z\backslash L) = G_L(x, y) \circ G_L(y, z),$$

and hence (SFD2) can be used.

Now assume $x \sim_{\mathscr{L},L} y$ and $z \in \Sigma^*$, i.e., $G_L(x, y) = (x\backslash L) \circ (y\backslash L) \in \mathscr{L}$. By (SFD3) $z\backslash[(x\backslash L) \circ (y\backslash L)] = (z\backslash(x\backslash L)) \circ (z\backslash(y\backslash L)) = (xz\backslash L) \circ (yz\backslash L) = G_L(xz, yz) \in \mathscr{L}$. Hence $xz \sim_{\mathscr{L},L} yz$.

(The rules for $\circ$, which are used here will be treated in the next section.)

The next theorem can be seen as a conversion of Theorem 2.4:

THEOREM 2.5. *If* $|\Sigma| \geqslant 2$ *and* $\sim_{\mathscr{L},L}$ *is a right congruence for all* L, *then* $\mathscr{L}$ *is an* SFD.

Proof. First the following will be shown: Let A, B be given. Then x, y, z, L can be found, such that

$$(x\backslash L) \circ (y\backslash L) = A \qquad \text{and} \qquad (y\backslash L) \circ (z\backslash L) = B.$$

Let $x = a$, $y = \epsilon$, $z = b$. The language L is recursively defined by:

$$\epsilon \notin L,$$
$$\sigma \in \Sigma - \{a, b\},\ w \in \Sigma^* \Rightarrow \sigma w \notin L$$
$$aw \in L :\Leftrightarrow [w \in A \Leftrightarrow w \in L]$$
$$bw \in L :\Leftrightarrow [w \in B \Leftrightarrow w \in L].$$

It is not hard to verify the desired properties.

If (SFD1) does not hold, reflexivity is missing.

If (SFD2) does not hold, i.e., $A, B \in \mathscr{L}$ and $A \circ B \notin \mathscr{L}$, define x, y, z, L as above. Then $x \sim_{\mathscr{L},L} y$, $y \sim_{\mathscr{L},L} z$ but not $x \sim_{\mathscr{L},L} z$.

If (SFD3) does not hold then A, z exist, such that $A \in \mathscr{L}$, $z\backslash A \notin \mathscr{L}$. Define x, y, L such that $(x\backslash L) \circ (y\backslash L) = A$. Consequently $x \sim_{\mathscr{L},L} y$ but not $xz \sim_{\mathscr{L},L} yz$.

It seems to be of a certain interest to take in consideration filters in this context though the filter axioms are stronger than it is necessary; filters are a convenient concept.

Similar to the proof of Theorem 2.4 is the demonstration of

THEOREM 2.6. *If* $\mathscr{L}$ *is an* SFD *then by*

$$A \sim_{\mathscr{L}} B :\Leftrightarrow A \circ B \in \mathscr{L}$$

an equivalence relation is defined.

Proof. For sake of clarity it will be shown that $A \sim_{\mathscr{L}} B$, $B \sim_{\mathscr{L}} C$ implies $A \sim_{\mathscr{L}} C$.

By the assumptions $A \circ B \in \mathscr{L}$, $B \circ C \in \mathscr{L}$ hold. Hence by (SFD2) $(A \circ B) \circ (B \circ C) = A \circ (B \circ B) \circ C = A \circ \Sigma^* \circ C = A \circ C \in \mathscr{L}$, i.e., $A \sim_{\mathscr{L}} B$.

THEOREM 2.7. *Let be* $\mathscr{L}$ *an* SFD. *If* $A \sim_{\mathscr{L}} B$ *then*

$$\sim_{\mathscr{L},A} = \sim_{\mathscr{L},B} .$$

Proof. By symmetry it is sufficient to show that $x \sim_{\mathscr{L},A} y$ implies that $x \sim_{\mathscr{L},B} y$.

From $A \circ B \in \mathscr{L}$ follows

$$x\backslash(A \circ B) = (x\backslash A) \circ (x\backslash B) \in \mathscr{L}.$$

By the assumption $(x\backslash A) \circ (y\backslash A) \in \mathscr{L}$. Thus

$$(x\backslash B) \circ (x\backslash A) \circ (x\backslash A) \circ (y\backslash A) = (x\backslash B) \circ \Sigma^* \circ (y\backslash A) = (x\backslash B) \circ (y\backslash A) \in \mathscr{L}.$$

A similiar argumentation gives

$$y\backslash(A \circ B) = (y\backslash A) \circ (y\backslash B) \in \mathscr{L},$$

and therefore

$$(x\backslash B) \circ (y\backslash A) \circ (y\backslash A) \circ (y\backslash B) = (x\backslash B) \circ (y\backslash B) \in \mathscr{L}.$$

(See the next section concerning the rules for $\circ$.)

3. PROPERTIES OF *FD*'S AND *SFD*'S

Defining SFD's it is sufficient to substitute (SFD1) by the weaker one

(SFD1′) $\mathscr{L} \neq \varnothing$,

since from $A \in \mathscr{L}$ follows $A \circ A = \Sigma^* \in \mathscr{L}$.

It is well-known that $(\mathfrak{P}(\Sigma^*), \triangle, \cap)$ forms a ring. The valid laws can be reformulated in terms of $\circ$:

$$A \triangle \varnothing = A, \qquad \text{therefore} \quad A \circ \varnothing = A^c$$
$$A \triangle A = \varnothing, \qquad \text{therefore} \quad A \circ A = \Sigma^*$$
$$A \triangle A^c = \Sigma^*, \qquad \text{therefore} \quad A \circ A^c = \varnothing.$$

$(A \triangle B)^c = A \triangle B^c$ implies $A \circ B = A \triangle B^c$. Thus $A \circ \Sigma^* = A \triangle \varnothing = A$. Hence

$$A \circ (B \circ C) = A \circ (B \triangle C^c) = A^c \triangle B \triangle C^c$$
$$= (A \circ B) \triangle C^c = (A \circ B) \circ C.$$

Therefore $(\mathfrak{P}(\Sigma^*), \circ)$ is a group, Σ^* being the unit and each element being self-inverse.

$$\begin{aligned}(A \circ B) \cup C &= [(A \circ B) \cup C]^{cc} = [(A \circ B)^c \cap C^c]^c = [(A \triangle B^c)^c \cap C^c]^c \\ &= [(A^c \triangle B^c) \cap C^c]^c = [(A^c \cap C^c) \triangle (B^c \cap C^c)]^c \\ &= [(A \cup C)^c \triangle (B \cup C)^c]^c = (A \cup C)^c \triangle (B \cup C) \\ &= (A \cup C) \circ (B \cup C),\end{aligned}$$

thus $(\mathfrak{P}(\Sigma^*), \circ, \cup)$ is a ring.

It is evident that $z \backslash (A \circ B) = (z \backslash A) \circ (z \backslash B)$ holds. For fixed z the mapping $A \mapsto z \backslash A$ is an endomorphism of rings.

It is possible to speak of the SFD generated by $\mathscr{L}$, since $\mathfrak{P}(\Sigma^*)$ is an SFD and arbitrary meets of SFD's are again SFD's.

Now some items to the FD's.

From $\varnothing \in \mathscr{L}$ follows $\mathscr{L} = \mathfrak{P}(\Sigma^*)$ if $\mathscr{L}$ is a filter. Therefore especially those FD's are of interest for which $\varnothing \in \mathscr{L}$ does not hold; call them *proper*.

Again it is possible to speak of the FD generated by $\mathscr{L}$, and it is interesting, whether or not it is proper.

EXAMPLE 1. Let be $\mathscr{D} = \{L \mid L^c \text{ is finite}\}$, i.e., $\mathscr{D}$ is the family of cofinite languages over Σ. It is not hard to see that $\mathscr{D}$ is an FD.

If $\Sigma = \{a\}$, it is possible to see a subset of a^* as a 0–1-sequence if one identifies the set with its characteristic function.

As an example, the set $a(aaa)^*$ corresponds to the 0–1-sequence $01001001001 \cdots$.

In the sequel k consecutive 1's in a 0–1-sequence are called *1-block of length k*.

THEOREM 3.1. *Let $\mathscr{L}' \subseteq \mathfrak{P}(a^*)$. If $\mathscr{L}'$ contains an A with the property that only 1-blocks with a length $\leqslant k$ appear, then the* FD $\mathscr{L}$ *generated by $\mathscr{L}'$ is not proper.*

Proof. Consider $(a \backslash A) \cap A$; this set is in $\mathscr{L}$ and contains only 1-blocks with a length $\leqslant k - 1$. Thus

$$\varnothing = A \cap (a \backslash A) \cap \cdots \cap (a^k \backslash A) \in \mathscr{L}.$$

The following theorem is a kind of conversion.

THEOREM 3.2. *If A contains arbitrary long 1-blocks, then the* FD $\mathscr{L}$ *generated by $\{A\}$ is proper.*

Proof. It is sufficient to show that it is impossible that sets which are in $\mathscr{L}$ by means of (FD2) and (FD4) are the empty set.

Hence it is sufficient to show that always

$$(a^{i_0}\backslash A) \cap \cdots \cap (a^{i_n}\backslash A) \neq \varnothing .$$

Thus it is sufficient to verifie that for all n

$$(a^0\backslash A) \cap \cdots \cap (a^n\backslash A) \neq \varnothing .$$

This is guaranteed by the existence of arbitrary long 1-blocks.

In order to generalize this interpretation as a sequence the following definition is given:

DEFINITION 3.3. $\Omega = \Omega(\Sigma) = \{(\omega_n)_{n=0}^{\infty} \mid \omega_{n+1} = \omega_n\sigma, \sigma \in \Sigma, \omega_0 = \epsilon\}$.
This leads to

EXAMPLE 2. $\mathscr{L} = \{L \mid \liminf_{n\to\infty} \mid \omega_n \cap L \mid/(n+1) = 1$ for all $\omega \in \Omega\}$ is a proper FD and $\mathscr{D} \subsetneqq \mathscr{L}$.

If lim inf is replaced by lim sup the generated FD $\mathscr{L}$ is not proper; let $\Sigma = \{a\}$ and construct A as follows: ($\omega = (0, 1, \ldots)$)

$$\text{the } n\text{-th 1-block is as large as } \frac{\mid A \cap \omega_n \mid}{n+1} \geqslant \frac{1}{n},$$

$$\text{the } n\text{-th 0-block is as large as } \frac{\mid A \cap \omega_n \mid}{n+1} \leqslant \frac{1}{n},$$

then A and A^c are in $\mathscr{L}$, and thus $\varnothing = A \cap A^c \in \mathscr{L}$.

It is impossible to dilate Theorem 3.1 for $\mid \Sigma \mid \geqslant 2$:

THEOREM 3.4. *Let be $\mathscr{L}' = \{\Sigma^* - Fa^* \mid F$ is a finite set$\}$ and $\mathscr{L} = \{L \mid$ there is an $L' \in \mathscr{L}'$ and $L' \subseteq L\}$. ($a \in \Sigma$ fixed.)*

Then $\mathscr{L}$ is a proper FD *and there is an $\omega \in \Omega$ such that*

$$\limsup_{n\to\infty} \frac{\mid L \cap \omega_n \mid}{n+1} = 1 \qquad \textit{for all} \quad L \in \mathscr{L}$$

does not hold.

Proof. First it is clear that $\Sigma^* - Fa^*$ can never be $\varnothing$.

It will be shown that for all finite sets F_1, F_2 there exists a finite set F_3, such that

$$(\Sigma^* - F_1a^*) \cap (\Sigma^* - F_2a^*) \supseteq \Sigma^* - F_3a^*$$

is valid. This is equivalent to

$$F_1a^* \cup F_2a^* \subseteq F_3a^*.$$

It is sufficient to choose $F_3 = F_1 \cup F_2$.

Now let F_1 be finite and $z \in \Sigma^*$. It will be shown that there exists a finite F_2 such that

$$z\backslash(\Sigma^* - F_1a^*) \supseteq \Sigma^* - F_2a^*$$

holds. This means

$$\Sigma^* - z\backslash(F_1a^*) \supseteq \Sigma^* - F_2a^*$$

or

$$z\backslash(F_1a^*) \subseteq F_2a^*.$$

It is possible to choose $F_2 = (z\backslash F_1) \cup \{\epsilon\}$, since from $w \in z\backslash(F_1a^*)$ follows that $zw \in F_1a^*$. The first case is $w = w_1w_2$ and $zw_1 \in F_1$, thus $w_1 \in z\backslash F_1$; the second one is $z = z_1a^k$ and $w = a^l$, thus $w \in a^*$.

Let be $\omega = (\epsilon, a, a^2, ...)$ and $L = \Sigma^* - a^* \in \mathscr{L}$. Then

$$\limsup_{n\to\infty} \frac{|\omega_n \cap L|}{n+1} = \limsup_{n\to\infty} 0 = 0.$$

This causes an Example 3.

4. Congruence Relations and Filters

The syntactic congruence $\approx_L$ (cf. Eilenberg (1974)) can be defined as follows:

$$x \approx_L y :\Leftrightarrow \text{for all } u \quad ux \in L \Leftrightarrow uy \in L \quad \text{and}$$
$$\text{for all } v \quad xv \in L \Leftrightarrow yv \in L.$$

This will be generalized in the sequel.

Definition 4.1. A filter (semifilter) $\mathscr{L}$ is called FD′ (SFD′) if it fulfills additionally

$$A \in \mathscr{L},\ z \in \Sigma^* \Rightarrow A/z \in \mathscr{L}.$$

Example. $\mathscr{D}$ is an FD′.

Definition 4.2. Let $\mathscr{L}_1$ be an SFD′, $\mathscr{L}_2$ an SFD:

$$x \approx_{\mathscr{L}_1,\mathscr{L}_2,L} y :\Leftrightarrow \text{for all } v \quad \text{holds } \{u \mid uxv \in L \Leftrightarrow uyv \in L\} \in \mathscr{L}_1 \quad \text{and}$$
$$\text{for all } u \quad \text{holds } \{u \mid uxv \in L \Leftrightarrow uyv \in L\} \in \mathscr{L}_2.$$

THEOREM 4.3. *Under the above mentioned assumptions* $\approx_{\mathscr{L}_1,\mathscr{L}_2,L}$ *is a congruence relation.*

Proof. The proof that $\approx_{\mathscr{L}_1,\mathscr{L}_2,L}$ is an equivalence relation corresponds to the proof of Theorem 2.4.

Assume $x \approx_{\mathscr{L}_1,\mathscr{L}_2,L} y$. It must be shown that for arbitrary s, t $sx \approx_{\mathscr{L}_1,\mathscr{L}_2,L} sy$ and $xt \approx_{\mathscr{L}_1,\mathscr{L}_2,L} yt$. By symmetrical argumentations it is sufficient to prove the second part. Let v be arbitrarily chosen. $\{u \mid uxtv \in L \Leftrightarrow uytv \in L\} \in \mathscr{L}_1$, since this holds for all v, especially for tv.

Now let u be arbitrarily chosen. $\{v \mid uxtv \in L \Leftrightarrow uytv \in L\} = t\backslash\{v \mid uxv \in L \Leftrightarrow uyv \in L\} \in \mathscr{L}_2$.

5. A Generalization of Regular Sets

It is natural to give the following

DEFINITION 5.1. Let $\mathscr{L}_1$ be an SFD′ and let $\mathscr{L}_2$ be an SFD. Define $\mathscr{R}_{\mathscr{L}_1,\mathscr{L}_2}$ to be the family of all formal languages L, such that $\approx_{\mathscr{L}_1,\mathscr{L}_2,L}$ has a finite index.

$\mathscr{R}_{\mathscr{L}_2}$ is the family of all L such that $\sim_{\mathscr{L}_2,L}$ has a finite index.

Obviously the following holds: If $\mathscr{L}_1 \subseteq \mathscr{L}'_1$, $\mathscr{L}_2 \subseteq \mathscr{L}'_2$ then $\mathscr{R}_{\mathscr{L}_1,\mathscr{L}_2} \subseteq \mathscr{R}_{\mathscr{L}'_1,\mathscr{L}'_2}$ and $\mathscr{R}_{\mathscr{L}_2} \subseteq \mathscr{R}_{\mathscr{L}'_2}$.

THEOREM 5.2. $\mathscr{R}_{\mathfrak{P}(\Sigma^*),\mathscr{L}_2} = \mathscr{R}_{\mathscr{L}_2}$.

Proof. The inclusion "$\subseteq$" is clear.

Now let (Q, Σ, δ, q_0) be the finite automaton without final states corresponding to $\sim_{\mathscr{L}_2,L}$. Furthermore let

$$\alpha: \Sigma^* \to Q^Q \quad \text{be defined by}$$
$$\alpha(w): q \mapsto \delta(q, w).$$

The congruence $\approx$ corresponding to the homomorphism α is a refinement of $\sim_{\mathscr{L}_2,L}$ and has a finite index.

Now assume $w \approx x$, i.e., $\alpha(w) = \alpha(x)$ and let u be an arbitrary element. Then $\alpha(uw) = \alpha(ux)$, i.e., $\delta(q_0, uw) = \delta(q_0, ux)$, thus $uw \sim_{\mathscr{L}_2,L} ux$, hence $\{v \mid uwv \in L \Leftrightarrow uxv \in L\} \in \mathscr{L}_2$; this means $w \approx_{\mathfrak{P}(\Sigma^*),\mathscr{L}_2,L} x$. Therefore $\approx_{\mathfrak{P}(\Sigma^*),\mathscr{L}_2,L}$ has at most as many classes as $\approx$, i.e., only a finite number of classes.

In the sequel it will be assumed that $\mathscr{L}_1$ is a FD′ and $\mathscr{L}_2$ is a FD.

LEMMA 5.3. *If L is $\mathscr{L}_1$, $\mathscr{L}_2$-regular then L^c is also $\mathscr{L}_1$, $\mathscr{L}_2$-regular.*

Proof. Obvious.

LEMMA 5.4. *If A, B are $\mathscr{L}_1$, $\mathscr{L}_2$-regular then $A \cap B$ is also $\mathscr{L}_1$, $\mathscr{L}_2$-regular.*

Proof. Define $\approx$ by $\approx_{\mathscr{L}_1,\mathscr{L}_2,A} \cap \approx_{\mathscr{L}_1,\mathscr{L}_2,B}$.
Let $x \approx y$ and u an arbitrary element:

$$(ux\backslash A) \circ (uy\backslash A) \in \mathscr{L}_2 \qquad \text{and} \qquad (ux\backslash B) \circ (uy\backslash B) \in \mathscr{L}_2 .$$

Therefore

$$[(ux\backslash A) \circ (uy\backslash A)] \cap [(ux\backslash B) \circ (uy\backslash B)] \in \mathscr{L}_2 ,$$

and this is a subset of

$$(ux\backslash A \cap B) \circ (uy\backslash A \cap B) = [(ux\backslash A) \cap (ux\backslash B)] \circ [(uy\backslash A) \cap (uy\backslash B)],$$

from which follows that the last set is in $\mathscr{L}_2$.

Symmetrically one gets for arbitrary v

$$(A \cap B/xv) \circ (A \cap B/yv) \in \mathscr{L}_1 .$$

Thus $x \approx_{\mathscr{L}_1,\mathscr{L}_2,A\cap B} y$. Therefore $\approx_{\mathscr{L}_1,\mathscr{L}_2,A\cap B}$ has not more classes than $\approx$; this yields a finite number of classes.

COROLLARY 5.5. *If A, B are $\mathscr{L}_1$, $\mathscr{L}_2$-regular then $A \cup B$ is also $\mathscr{L}_1$, $\mathscr{L}_2$-regular.*

As a summary can be stated: ($\varnothing$ is $\mathscr{L}_1$, $\mathscr{L}_2$-regular).

THEOREM 5.6. *$\mathscr{R}_{\mathscr{L}_1,\mathscr{L}_2}$ is a boolean algebra.*

6. THE CASE $\mathscr{D}$

The case $\mathscr{D}$ seems to be the most interesting one, therefore some remarks concerning this filter will be presented.

If $\sim_{\mathscr{D},L}$ is of finite index, then it is possible to construct the corresponding finite automaton without final states.

It seems suggestive to believe that the following holds: If suitable final states are chosen, a formal language L', "being simpler as L and similar to L'" is obtained. But the following is possible: There are two infinite classes in the minimal automaton of L which coincide with respect to $\sim_{\mathscr{D},L}$. Exactly one of them is a final state; thus in very case $|L' \triangle L| = \infty$. This seems to be not very satisfactory.

It is even possible that this happens considering $\approx_L$ which is a refinement of $\sim_L$. The two classes coincide with respect to $\approx_{\mathfrak{P}(\Sigma^*),\mathscr{D},L}$. The language $c^*\{\epsilon, a\} \cup c^*\{aa, ba\}\, c^*$ yields an example.

Furthermore it is false to believe that it is impossible, that the automaton corresponding to $\sim_{\mathscr{D},L}$ contains finite classes; a counter-example is obtained by taking $L = ab^*$. (Two terminal symbols are necessary!)

To obtain the automaton without final states starting with the minimal automaton for L one can proceed as follows:

Assume $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ and let be given the question whether or not the classes x, y coincide with respect to $\sim_{\mathscr{D},L}$. One considers the expression

$$\delta(x, \sigma_1) = \delta(y, \sigma_1) \wedge \cdots \wedge \delta(x, \sigma_n) = \delta(y, \sigma_n)$$

and substitutes distinct x', y' by the analogous expression.

If there appears finally

$$x_1 = x_1 \wedge \cdots \wedge x_s = x_s,$$

the classes coincide, in the other case it happens that after some steps of replacement an expression $\bar{x} = \bar{y}$ will be obtained a second time (a "loop"). Then the classes do not coincide.

A subset $L \subseteq \Sigma^*$ is called *disjunctive* (Shyr, 1977) or *rigid* (Eilenberg, 1976, p. 187) if from $x \approx_L y$ follows $x = y$.

It is natural to give the following

DEFINITION 6.1. L is called $\mathscr{L}_1, \mathscr{L}_2$-disjunctive ($\mathscr{L}$-disjunctive), if $x \approx_{\mathscr{L}_1,\mathscr{L}_2,L} y$ ($x \sim_{\mathscr{L},L} y$) implies $x = y$.

THEOREM 6.2. *If a language L is $\{\Sigma^*\}$-disjunctive it is also $\mathscr{D}$-disjunctive.*

Proof. Let be $x \neq y$ and $a \in \Sigma$. Because $xa \not\sim_L ya$ holds there is a $z \in \Sigma^*$ such that exactly one of the elements xaz, yaz is in L. Thus there is a $z_1 \in \Sigma^+$ such that exactly one of xz_1, yz_1 is in L. Applying this argumentation to xz_1, yz_1 one obtains $z_2 \in \Sigma^+$ etc. Finally one gets an infinite set $\{z_1, z_2, \ldots\}$ such that for all i exactly one of xz_i, yz_i is in L. Thus $x \sim_{\mathscr{D},L} y$ is impossible.

The results discussed in this paper seem to be only a small part of problems which can be considered in this context. To give only one example the following open question is cited: Does $\mathscr{R} = \mathscr{R}_{\mathscr{D}}$ hold?

ACKNOWLEDGMENT

The author thanks the referee for several helpful remarks.

RECEIVED: March 2, 1979; REVISED: September 12, 1979

References

Bell, J. L., and Machover, M. (1977), "A Course in Mathematical Logic," North-Holland, Amsterdam/New York/Oxford.

Benda, V., and Bendova, K. (1976), Recognizable filters and ideals, *Comment. Math Univ. Carolinae* **17**, No. 2, 251–259.

Eilenberg, S. (1974), "Automata, Languages and Machines," Vol. A, Academic Press, New York/London.

Eilenberg, S. (1976), "Automata, Languages and Machines," Vol. B, Academic Press, New York/London.

Prodinger, H., and Urbanek, F. J. (1979), Language operators related to Init, *Theoret. Comput. Sci.* **8**, 161–175.

Prodinger, H. (1979), Topologies on free monoids induced by closure operators of a special type, to appear.

Shyr, H. J. (1977), Disjunctive languages on a free monoid, *Inform. Contr.* **34**, 123–129.